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Hochschild homology of complete intersections

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Abstract

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Let k be an arbitrary commutative ring. We compute the Hochschild homology $\mathrm{HH}_*(A, A)$ of the k-algebra $A = k[x_1, \ldots, x_n]/\langle f_1, \ldots, f_r\rangle$, when $f_1, \ldots, f_r$ is a regular sequence.

Introduction

The aim of this paper is the computation of the Hochschild homology and cohomology of the ring $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$, where k is an arbitrary commutative ring with 1 and $f_1, \ldots, f_r$ is a regular sequence in $k[X_1, \ldots, X_n]$.

The paper is divided in three sections.

In the first one a quick review of some basic notions in the Hochschild theory is made (see, for instance, [3, Chapter 10]).

In the second section we obtain a free resolution $R(A)$ of A as an A^e-module which is simpler than the one given by Hochschild, and we define a map h_* of $R(A)$ to the Hochschild resolution $(A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ that is an A^e-algebra map when $R(A)$ has a product which is naturally defined and $(A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ has the shuffle product shown in Section 1.

In the third section we compute the Hochschild homology $H_*(A, M)$ and cohomology $H^*(A, M)$ of A with coefficients in an A-module M, looked at as an A^e-module through the multiplication map $\mu : A^e \to A$. We obtain them as a function of the homology of the generalized Koszul complex of the given module. When $r = 1$, our results are the same as the ones in [1].

1. Preliminaries

1.1. Hochschild homology

Let A be an associative algebra with identity over a commutative ring k. We will use the abbreviation $\bar{A}^{\otimes^n}$ for the n-fold tensor product of $\bar{A}$ over k, where $\bar{A} = A/k$. Let $b' : A \otimes \bar{A}^{\otimes^n} \otimes A \to A \otimes \bar{A}^{\otimes^{n-1}} \otimes A$ denote the following map:

$$b'(a_0 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_{n+1} .$$

The chain complex

$$\cdots \xrightarrow{b'} A \otimes \bar{A}^{\otimes^n} \otimes A \xrightarrow{b'} A \otimes \bar{A}^{\otimes^{n-1}} \otimes A \xrightarrow{b'} \cdots \xrightarrow{b'} A \otimes \bar{A} \otimes A \xrightarrow{b'} A \otimes A \xrightarrow{b'} A ,$$

is the *standard normalized Hochschild resolution* $(A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ of A. It is acyclic because there exists a retraction homotopy

$$\varepsilon_0 : A \otimes \bar{A}^{\otimes^n} \otimes A \to A \otimes \bar{A}^{\otimes^{n+1}} \otimes A$$

defined by

$$\varepsilon_0(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1} .$$

Let M be a right module over $A^e = A \otimes A^{op}$. Upon tensoring the normalized Hochschild resolution $(A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ with M over A^e, we obtain the chain complex $(M \otimes \bar{A}^{\otimes^*}, b)$,

$$\cdots \xrightarrow{b} M \otimes \bar{A}^{\otimes^n} \xrightarrow{b} M \otimes \bar{A}^{\otimes^{n-1}} \xrightarrow{b} \cdots \xrightarrow{b} M \otimes \bar{A} \xrightarrow{b} M ,$$

where

$$\begin{aligned} b(m \otimes a_1 \otimes \cdots \otimes a_n) &= m(a_1 \otimes 1) \otimes a_2 \otimes \cdots \otimes a_n \\ &+ \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_{i-1} \otimes a_i a_{i+1} \otimes a_{i+2} \otimes \cdots \otimes a_n \\ &+ (-1)^n m(1 \otimes a_n) \otimes a_1 \otimes \cdots \otimes a_{n-1} , \end{aligned}$$

which we call the *normalized Hochschild complex*. Its homology is the *Hochschild homology* $H_*(A, M)$. On the other hand, applying $\operatorname{Hom}_{A^e}(\ , M)$ to the standard normalized Hochschild resolution $(A \otimes \bar{A}^{\otimes^*} \otimes A, b')$, we obtain the cochain complex

$$\begin{aligned}\mathrm{Hom}_{A^e}(A \otimes A, M) &\xrightarrow{\mathrm{Hom}(b',M)} \mathrm{Hom}_{A^e}(A \otimes \bar{A} \otimes A, M) \\ &\to \cdots \to \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes^{n-1}} \otimes A, M) \\ &\xrightarrow{\mathrm{Hom}(b',M)} \mathrm{Hom}_{A^e}(A \otimes \bar{A}^{\otimes^n} \otimes A, M) \to \cdots .\end{aligned}$$

Its homology is the *Hochschild cohomology* $H^*(A, M)$. When A is projective over k one has $H_n(A, M) = \mathrm{Tor}_n^{A^e}(A, M)$ and $H^n(A, M) = \mathrm{Ext}^n_{A^e}(A, M)$.

1.2. Shuffle product

From now on A is commutative. In this case, the normalized Hochschild resolution becomes a differential graded strictly anti-commutative A^e-algebra, where strict means that the square of any odd degree element is zero, with the product defined by

$$\begin{aligned}&(a \otimes a_1 \otimes \cdots \otimes a_p \otimes b) * (a' \otimes a_{p+1} \otimes \cdots \otimes a_{p+q} \otimes b') \\ &\quad = \sum_{B_{p,q}} \mathrm{sgn}(\sigma) aa' \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)} \otimes bb' ,\end{aligned}$$

where

$$\begin{aligned}B_{p,q} = \{\sigma \in S_{p+q} \colon \sigma(1) < \cdots < \sigma(p) \text{ and} \\ \sigma(p+1) < \cdots < \sigma(p+q)\} .\end{aligned}$$

Upon tensoring the normalized Hochschild resolution with A over A^e, we obtain the *shuffle product*

$$\begin{aligned}&(a \otimes a_1 \otimes \cdots \otimes a_p) * (a' \otimes a_{p+1} \otimes \cdots \otimes a_{p+q}) \\ &\quad = \sum_{B_{p,q}} \mathrm{sgn}(\sigma) aa' \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)} .\end{aligned}$$

Hence, the Hochschild homology $H_*(A) = H_*(A, A)$ is a graded strictly anti-commutative algebra over A.

2. The simplified resolution

2.1. Introduction

Let $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$, with k an arbitrary commutative ring with 1, and $f_1, \ldots, f_r$ a regular sequence in $k[X_1, \ldots, X_n]$.

In this section we obtain a free resolution $R(A)$ of A as an A^e-module. This resolution becomes a differential graded strictly anti-commutative A^e-algebra in a natural way. Moreover, we define explicitly a differential graded algebra morphism $h_*: R(A) \to (A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ which will be used in the next section.

Let $A' = k[X_1, \ldots, X_n]$. We shall use the Taylor series development $T: A' \to A'^e$ given by $T(p) = 1 \otimes p - p \otimes 1$, studied in [4]. We also freely use the structure of A'^e as A'-module by the action of A' on the first factor of A'^e, i.e. $a(b \otimes c) = ab \otimes c$. From the formula $T(PQ) = PT(Q) + QT(P) + T(P)T(Q)$, it is easy to see that, for every $P \in A'$, $T(P)$ can be written as a polynomial in $A'[T(X_1), \ldots, T(X_n)]$, where the coefficients in A', of each monomial in the $T(X_i)$'s, are the same as those of the standard Taylor series. We shall use the following definition:

2.1.1. Definition. Given $P \in k[X_1, \ldots, X_n]$, we shall call $T_j(P)$ the sum of the monomials of $T(P)$ which are multiples of $T(X_j)$ and not multiples of $T(X_i)$ for $i < j$, i.e.,

$$T_j(P) = \sum_{\substack{i_j \geq 1 \\ i_{j+1}, \ldots, i_n}} \frac{1}{i_j! \cdots i_n!} \cdot \frac{\partial^{\Sigma i_k} P}{\partial X_j^{i_j} \cdots \partial X_n^{i_n}} \cdot T(X_j)^{i_j} \cdots T(X_n)^{i_n}$$

$$(1 \leq j \leq n).$$

We shall state a technical proposition that will be used later.

2.1.2. Proposition. (a) T_j *is k-linear, for* $1 \leq j \leq n$.

(b) $T(P) = \Sigma_{j=1}^n T_j(P)$.

Proof. Trivial. □

2.2. The resolution $R(A)$

As before, let $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$, where k is an arbitrary commutative ring with 1, $f_1, \ldots, f_r$ is a regular sequence in $k[X_1, \ldots, X_n]$ and let $A^e = A \otimes_k A^{op}$.

Let $D(A)$ be the exterior algebra over A^e of the free A^e-module $A^e e_1 \oplus \cdots \oplus A^e e_n$ and $F(A)$ the algebra of divided powers over $D(A)$ with r variables $t_1, \ldots, t_r$. We recall that the algebra of divided powers with r variables over a ring D is a free module over D with basis $t_1^{(p_1)} \cdots t_r^{(p_r)}$ $(p_j \in \mathbb{N}_0)$ and the product given by

$$(t_1^{(p_1)} \cdots t_r^{(p_r)})(t_1^{(q_1)} \cdots t_r^{(q_r)}) = \prod_{k=1}^{r} \binom{p_k + q_k}{p_k} (t_1^{(p_1+q_1)} \cdots t_r^{(p_r+q_r)}) . \quad (1)$$

We shall assign degree 1 to the elements e_i and degree $2p$ to $t_j^{(p)}$. Hence $F(A)$ becomes a strictly anticommutative graded A^e-algebra.

We shall define a derivation d_* by

$$d_1(e_i) = T(X_i)\,, \qquad d_{2p}(t_j^{(p)}) = t_j^{(p-1)} d_2(t_j)$$

and

$$d_2(t_j) = \sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot e_i\,.$$

The algebra $F(A)$ with the derivation just defined will be called $R(A)$ and its homogeneous component of degree m will be called $R(A)_m$.

It is obvious that $R(A)_m$ is a free A^e-module with basis all combinations $e_{i_1 \cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)}$ ($s, p_1, \ldots, p_r \geq 0$ and $s + 2(p_1 + \cdots + p_r) = m$), where by $e_{i_1 \cdots i_s}$ we mean $e_{i_1} \wedge \cdots \wedge e_{i_s}$ $(1 \leq i_1 < \cdots < i_s \leq n)$.

2.2.1. Remark. Let $(x_i)_{1 \leq i \leq r}$ be a sequence of commuting elements of a ring B and let M be a B-module. We denote by $K_*(M, (x_i)_{1 \leq i \leq r})$ the Koszul complex

$$0 \to M^{\binom{r}{r}} \xrightarrow{d_{r-1}} M^{\binom{r}{r-1}} \xrightarrow{d_{r-2}} \cdots \xrightarrow{d_1} M^{\binom{r}{1}} \xrightarrow{d_0} M^{\binom{r}{0}} \longrightarrow 0\,,$$

where $M^{\binom{r}{m}}$ denotes the direct sum of $\binom{r}{m}$ copies of M indexed by $e_{i_1 \cdots i_m}$ $(1 \leq i_1 < \cdots < i_m \leq n)$ and

$$d_{m-1}(\vartheta \cdot e_{i_1 \cdots i_m}) = \sum_{j=1}^{m} (-1)^{j+1} \vartheta \cdot x_{i_j} e_{i_1 \cdots \hat{i}_j \cdots i_m}$$

for $\vartheta \in M$.

2.2.2. Remark. It is clear that the exterior algebra $D(A)$ with the derivation d (i.e., $d(e_i) = T(X_i)$) is the Koszul complex $K_*(A^e, T(X_i)_{1 \leq i \leq n})$, and $R(A)$ is obtained from it by killing the cycles

$$\sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot e_i \in A^{e\binom{n}{1}}$$

with the method given by Tate in [5].

At this point we might prove that $R(A)$ is an A^e-free resolution of A.

2.2.3. Lemma. *Let* $A' = k[X_1, \ldots, X_n]$ *and* $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$. *The morphism of differential graded algebras*

$$\xi_* : K_*(A', (f_1, \ldots, f_r, f_1, \ldots, f_r)) \to K_*(A. (f_1, \ldots, f_r))\,,$$

given by

$$\xi_1(e_i) = \begin{cases} e_i & \textit{if } i \le r\,, \\ 0 & \textit{if } i > r\,, \end{cases}$$

induces an isomorphism in homology.

Proof. Let B_{**} be the double complex

$$\begin{array}{ccccccccc}
0 & & 0 & & & & 0 & & 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
A'^{\binom{r}{r}} \otimes A'^{\binom{r}{0}} & \longleftarrow & A'^{\binom{r}{r}} \otimes A'^{\binom{r}{1}} & \longleftarrow \cdots \longleftarrow & & & A'^{\binom{r}{r}} \otimes A'^{\binom{r}{r-1}} & \longleftarrow & A'^{\binom{r}{r}} \otimes A'^{\binom{r}{r}} \longleftarrow 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
A'^{\binom{r}{r-1}} \otimes A'^{\binom{r}{0}} & \longleftarrow & A'^{\binom{r}{r-1}} \otimes A'^{\binom{r}{1}} & \longleftarrow \cdots \longleftarrow & & & A'^{\binom{r}{r-1}} \otimes A'^{\binom{r}{r-1}} & \longleftarrow & A'^{\binom{r}{r-1}} \otimes A'^{\binom{r}{r}} \longleftarrow 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
\vdots & & \vdots & & & & \vdots & & \vdots \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
A'^{\binom{r}{1}} \otimes A'^{\binom{r}{0}} & \longleftarrow & A'^{\binom{r}{1}} \otimes A'^{\binom{r}{1}} & \longleftarrow \cdots \longleftarrow & & & A'^{\binom{r}{1}} \otimes A'^{\binom{r}{r-1}} & \longleftarrow & A'^{\binom{r}{1}} \otimes A'^{\binom{r}{r}} \longleftarrow 0 \\
\downarrow & & \downarrow & & & & \downarrow & & \downarrow \\
A'^{\binom{r}{0}} \otimes A'^{\binom{r}{0}} & \longleftarrow & A'^{\binom{r}{0}} \otimes A'^{\binom{r}{1}} & \longleftarrow \cdots \longleftarrow & & & A'^{\binom{r}{0}} \otimes A'^{\binom{r}{r-1}} & \longleftarrow & A'^{\binom{r}{0}} \otimes A'^{\binom{r}{r}} \longleftarrow 0
\end{array}$$

whose p,q entry is $A'^{\binom{r}{p}} \otimes A'^{\binom{r}{q}}$, the horizontal maps are those of the Koszul complex $K_*(A', (f_1, \dots, f_r))$, and the vertical maps are $(-1)^p$ times those of the above Koszul complex.

Since $f_1, \dots, f_r$ is a regular sequence, the map

$$\bar{\xi}_* : \operatorname{Tot}(B_{**}) \to K_*(A, (f_1, \dots, f_r))\,,$$

given by

$$\bar{\xi}_i(e_{j_1 \cdots j_i} \otimes 1) = e_{j_1 \cdots j_i}\,,$$

and

$$\bar{\xi}_i(e_{j_1 \cdots j_s} \otimes e_{j'_1 \cdots j'_{i-s}}) = 0 \quad \text{if } s < i\,,$$

is a quasi-isomorphism. We finish the proof by observing that there is a canonical isomorphism $K_*(A', (f_1, \dots, f_r, f_1, \dots, f_r)) \to \operatorname{Tot}(B_{**})$ such that

$$\begin{array}{ccc}
K_*(A', (f_1, \dots, f_r, f_1, \dots, f_r)) & \longrightarrow & \operatorname{Tot}(B_{**}) \\
\big\downarrow = & & \big\downarrow \bar{\xi}_* \\
K_*(A', (f_1, \dots, f_r, f_1, \dots, f_r)) & \xrightarrow{\xi_*} & K_*(A, (f_1, \dots, f_r))
\end{array}$$

is commutative. $\square$

2.2.4. Lemma. *Let $A' = k[X_1, \ldots, X_n]$ and $A = A'/\langle f_1, \ldots, f_r\rangle$. The homology of the Koszul complex $K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})$ is*

$$H_m K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})) = A^{\mathrm{e}\binom{r}{m}} \quad \textit{for } 0 \le m \le r$$

and

$$H_m(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})) = 0 \quad \textit{for } m > r\,.$$

Moreover, $1\otimes 1 \in A^{\mathrm{e}\binom{n}{0}}$ freely generates $H_0(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n}))$ and the elements of the form

$$\left(\sum_{i=1}^{n} \frac{T_i(f_{j_1})}{T(X_i)} \cdot e_i\right) \wedge \cdots \wedge \left(\sum_{i=1}^{n} \frac{T_i(f_{j_m})}{T(X_i)} \cdot e_i\right) \in A^{\mathrm{e}\binom{r}{m}}$$
$$(1 \le j_1 < \cdots < j_m \le r)\,,$$

which are distinct and nonzero, freely generate $H_m(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n}))$.

Proof. Since $T(X_i)_{1\le i\le n}$ is a regular sequence in A'^{e}, the complex

$$0 \to A'^{\mathrm{e}\binom{n}{n}} \xrightarrow{d_{n-1}} A'^{\mathrm{e}\binom{n}{n-1}} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} A'^{\mathrm{e}\binom{n}{1}} \xrightarrow{d_0} A'^{\mathrm{e}\binom{n}{0}} \xrightarrow{\mu} A'\,,$$

where

$$0 \to A'^{\mathrm{e}\binom{n}{n}} \xrightarrow{d_{n-1}} A'^{\mathrm{e}\binom{n}{n-1}} \xrightarrow{d_{n-2}} \cdots \xrightarrow{d_1} A'^{\mathrm{e}\binom{n}{1}} \xrightarrow{d_0} A'^{\mathrm{e}\binom{n}{0}} \to 0\,,$$

is the Koszul complex $K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n})$ and μ, given by $\mu(a\otimes b) = ab$, is an A'^{e}-resolution of A'. Since $K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n}) \otimes_{A'^{\mathrm{e}}} A^{\mathrm{e}}$ is canonically isomorphic to $K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})$,

$$H_*(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})) = \mathrm{Tor}_*^{A'^{\mathrm{e}}}(A', A^{\mathrm{e}})\,.$$

Hence, in order to compute this homology, we can use the A'^{e}-resolution of A^{e},

$$0 \longrightarrow A'^{\mathrm{e}\binom{2r}{2r}} \xrightarrow{d'_{2r-1}} A'^{\mathrm{e}\binom{2r}{2r-1}} \xrightarrow{d'_{2r-2}} \cdots$$
$$\xrightarrow{d'_1} A'^{\mathrm{e}\binom{2r}{1}} \xrightarrow{d'_0} A'^{\mathrm{e}\binom{2r}{0}} \xrightarrow{\pi} A^{\mathrm{e}}\,,$$

obtained using the Koszul complex $K_*(A'^{\mathrm{e}}, (f_1\otimes 1, \ldots, f_r\otimes 1, 1\otimes f_1, \ldots, 1\otimes f_r))$ and the canonical projection $\pi : A'^{\mathrm{e}\binom{2r}{0}} \to A^{\mathrm{e}}$.

Now, we can consider the diagram of quasi-isomorphisms

$$\begin{tikzcd}[column sep=tiny]
& K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n}) \otimes_{A'^{\mathrm{e}}} K_*(A'^{\mathrm{e}}, (f_1\otimes 1, \ldots, f_r\otimes 1, 1\otimes f_1, \ldots, 1\otimes f_r)) \arrow[dl, "\psi^1_*"'] \arrow[dr, "\varphi^1_*"] & \\
K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n}) \otimes_{A'^{\mathrm{e}}} A^{\mathrm{e}} \arrow[d, "\psi^2_*"] & & A' \otimes_{A'^{\mathrm{e}}} K_*(A'^{\mathrm{e}}, (f_1\otimes 1, \ldots, f_r\otimes 1, 1\otimes f_1, \ldots, 1\otimes f_r)) \arrow[d, "\varphi^2_*"] \\
K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n}) & & K_*(A', (f_1, \ldots, f_r, f_1, \ldots, f_r)) \arrow[d, "\xi_*"] \\
& & K_*(A, (f_1, \ldots, f_r))
\end{tikzcd}$$

where ξ_* is the morphism defined in Lemma 2.2.3 and ψ^1_*, ψ^2_*, φ^1_* and φ^2_* are the canonic maps. From this diagram it follows that

$$H_m(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})) = A^{\mathrm{e}\binom{r}{m}} \quad \text{for } 0 \le m \le r$$

and

$$H_m(K_*(A^{\mathrm{e}}, T(X_i)_{1\le i\le n})) = 0 \quad \text{for } m > r\,.$$

Now, it is easy to see that $\varphi_* = \xi_* \circ \varphi^2_* \circ \varphi^1_*$ and $\psi_* = \psi^2_* \circ \psi^1_*$ are both morphisms of differential graded algebras. So, in order to finish the proof it is sufficient to note that $\alpha = (1\otimes 1)\otimes(1\otimes 1) \in A'^{\mathrm{e}} \otimes_{A'^{\mathrm{e}}} A'^{\mathrm{e}}$ is a cycle of degree zero in

$$K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n}) \\ \otimes_{A'^{\mathrm{e}}} K_*(A'^{\mathrm{e}}, (f_1\otimes 1, \ldots, f_r\otimes 1, 1\otimes f_1, \ldots, 1\otimes f_r))$$

and verify that $\psi_0(\alpha) = 1\otimes 1 \in A^{\mathrm{e}}$ and $\varphi_0(\alpha) = 1 \in A$. Similarly,

$$\beta_j = (1\otimes 1)\otimes(e_j - e_{j+r}) + \sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot e_i \otimes (1\otimes 1) \\ \in (A'^{\mathrm{e}} \otimes_{A'^{\mathrm{e}}} A'^{\mathrm{e}\binom{2r}{1}}) \oplus (A'^{\mathrm{e}\binom{n}{1}} \otimes_{A'^{\mathrm{e}}} A'^{\mathrm{e}}) \quad (1 \le j \le r)$$

is a cycle of degree one in

$$K_*(A'^{\mathrm{e}}, T(X_i)_{1\le i\le n}) \\ \otimes_{A'^{\mathrm{e}}} K_*(A'^{\mathrm{e}}, (f_1\otimes 1, \ldots, f_r\otimes 1, 1\otimes f_1, \ldots, 1\otimes f_r))$$

whence

$$\psi_1(\beta_j) = \sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot e_i \in A^{\mathrm{e}\binom{n}{1}} \quad \text{and} \quad \varphi_1(\beta_j) = e_j \in A^{\binom{2r}{1}}\,. \qquad \square$$

2.2.5. Lemma. *Let $0 \le u \le r$ and $R^u(A)$ be the complex*

$$\cdots \longrightarrow A_m^u \xrightarrow{d_{m-1}^u} A_{m-1}^u \xrightarrow{d_{m-2}^u} \cdots \xrightarrow{d_1^u} A_1^u \xrightarrow{d_0^u} A_0^u ,$$

where A_m^u is the A^e-free module with basis

$$e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_u^{(p_u)} = e_{i_1} \wedge \cdots \wedge e_{i_s} t_1^{(p_1)} \cdots t_u^{(p_u)}$$
$$(1 \le i_1 < \cdots < i_s \le n;\ s, p_1, \ldots, p_u \ge 0 \text{ and }$$
$$s + 2(p_1 + \cdots + p_u) = m)$$

and, with the convention of $t_j^{(p)} = 0$ if $p < 0$,

$$d_{m-1}^u(e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_u^{(p_u)})$$
$$= \sum_{k=1}^{s} (-1)^{k+1} T(X_{i_k}) e_{i_1\cdots \hat{i}_k \cdots i_s} t_1^{(p_1)} \cdots t_u^{(p_u)}$$
$$+ \sum_{j=1}^{u} \Big(\sum_{k=1}^{n} \frac{T_k(f_j)}{T(X_k)} \cdot e_k \wedge e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_j^{(p_j - 1)} \cdots t_u^{(p_u)} \Big) .$$

We assert that

$$H_m(R^u(A)) = A^{\binom{r-u}{m}} \quad \text{for } 0 \le m \le r-u$$

and

$$H_m(R^u(A)) = 0 \quad \text{for } m > r-u .$$

Moreover, $1 \otimes 1 \in A^{e\binom{n}{0}}$ freely generates $H_0(K_*(A^e, T(X_i)_{1 \le i \le n}))$ and the elements of the form

$$\Big(\sum_{i=1}^{n} \frac{T_i(f_{j_1})}{T(X_i)} \cdot e_i \Big) \wedge \cdots \wedge \Big(\sum_{i=1}^{n} \frac{T_i(f_{j_m})}{T(X_i)} \cdot e_i \Big) \in A^{e\binom{r-u}{m}}$$
$$(1 + u \le j_1 < \cdots < j_m \le r) ,$$

which are distinct and nonzero, freely generate $H_m(K_*(A^e, T(X_i)_{1 \le i \le n}))$ for $1 \le m \le r-u$.

Proof. It will be done by induction on u. For $u = 0$ we can use Lemma 2.2.4. Suppose that the assertion is true for u and that $u < r$. $R^{u+1}(A)$ is the total complex of the double complex

$$\begin{array}{ccccccc}
\downarrow d^u_3 & & \downarrow d^u_2 & & \downarrow d^u_1 & & \downarrow d^u_0 \\
A^u_3 & \xleftarrow{d^h_3} & A^u_2 \cdot t^{(1)}_{u+1} & \xleftarrow{d^h_2} & A^u_1 \cdot t^{(2)}_{u+1} & \xleftarrow{d^h_1} & A^u_0 \cdot t^{(3)}_{u+1} \\
\downarrow d^u_2 & & \downarrow d^u_1 & & \downarrow d^u_0 & & \\
A^u_2 & \xleftarrow{d^h_2} & A^u_1 \cdot t^{(1)}_{u+1} & \xleftarrow{d^h_1} & A^u_0 \cdot t^{(2)}_{u+1} & & \\
\downarrow d^u_1 & & \downarrow d^u_0 & & & & \\
A^u_1 & \xleftarrow{d^h_1} & A^u_0 \cdot t^{(1)}_{u+1} & & & & \\
\downarrow d^u_0 & & & & & & \\
A^u_0 & & & & & &
\end{array}$$

where $d^h_m : A^u_{m-1} \cdot t^{(p+1)}_{u+1} \to A^u_m \cdot t^{(p)}_{u+1}$ is defined by

$$\begin{aligned}
&d^h_m(e_{i_1 \cdots i_s} t_1^{(p_1)} \cdots t_u^{(p_u)} t_{u+1}^{(p+1)})) \\
&\quad = \sum_{i=1}^{n} \frac{T_i(f_{u+1})}{T(X_i)} \cdot e_i \wedge e_{i_1 \cdots i_s} t_1^{(p_1)} \cdots t_u^{(p_u)} t_{u+1}^{(p)} \\
&\quad (s + 2(p_1 + \cdots + p_u) = m - 1)\,.
\end{aligned}$$

From the inductive hypothesis it is easily seen that the second complex $E^2_{i,j}$, obtained by computing first the homologies with respect to the differentiation d^u, and then those with respect to the differentiation d^h is

$$E^2_{i,j} = \begin{cases} A^{e\binom{r-u}{i}} & \text{if } j = 0 \text{ and } 0 \le i \le r - u - 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

and that $1 \otimes 1 \in A^{e\binom{n}{0}}$ freely generates $E^2_{0,0}$ and the elements of the form

$$\left(\sum_{k=1}^{n} \frac{T_k(f_{j_1})}{T(X_k)} \cdot e_k\right) \wedge \cdots \wedge \left(\sum_{k=1}^{n} \frac{T_k(f_{j_i})}{T(X_k)} \cdot e_k\right) \in A^{e\binom{r-u}{i}}$$
$$(1 + u \le j_1 < \cdots < j_i \le r)\,,$$

which are distinct and nonzero, freely generate $E^2_{i,j}$. The proof follows immediately. □

2.2.6. Theorem. *$R(A)$ is an A^e-free resolution of A.*

Proof. It is easy to see that $R(A) = R^r(A)$. The assertion follows immediately applying Lemma 2.2.5 for $u = r$. □

2.3. Construction of h_*

Now we will construct a morphism $h_* : R(A) \to (A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ of differential graded strictly anti-commutative algebras.

In order to define h_* we need several properties of the shuffle product that was defined in Section 1.2 for the Hochschild resolution of a commutative k-algebra A. These properties have been studied in [1, Section 2].

2.3.1. Definition. Given

$$\alpha = a\otimes a_1 \otimes \cdots \otimes a_p \otimes b \in A\otimes \bar{A}^{\otimes p}\otimes A$$

and

$$\beta = a'\otimes a_{p+1} \otimes \cdots \otimes a_{p+q} \otimes b' \in A\otimes \bar{A}^{\otimes q}\otimes A\ ,$$

we can define

$$\alpha \square \beta = \sum_{B'_{p,q}} \operatorname{sgn}(\sigma) aa' \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(p+q)} \otimes bb'\ ,$$

where

$$B'_{p,q} = \{\sigma \in B_{p,q} \colon \sigma(1) < \sigma(p+1)\} = \{\sigma \in B_{p,q} \colon \sigma(1) = 1\}\ .$$

2.3.2. Remark. It is immediate that $\square$ has the following properties:

(a) $\alpha * \beta = \alpha \square \beta + (-1)^{pq} \beta \square \alpha$, for $\alpha \in A\otimes \bar{A}^{\otimes p}\otimes A$ and $\beta \in A\otimes \bar{A}^{\otimes q}\otimes A$ (in consequence, $*$ is strictly anti-commutative).

(b) $(\alpha_1 \square \alpha_2)\square \alpha_3 = \alpha_1 \square (\alpha_2 \square \alpha_3) + (-1)^{p_2p_3}\alpha_1 \square (\alpha_3 \square \alpha_2)$ for $\alpha_i \in A\otimes \bar{A}^{\otimes p_i}\otimes A$.

(c) $\varepsilon_0(\alpha * \varepsilon_0(\beta)) = \varepsilon_0(\alpha) \square \varepsilon_0(\beta)$.

2.3.3. Definition. Given $\alpha \in A\otimes \bar{A}^{\otimes p}\otimes A$, we define $\alpha^{(n)}$ for $n \geq 0$ by induction in the following way:

$$\alpha^{(0)} = 1\ , \qquad \alpha^{(n+1)} = \alpha \square \alpha^{(n)}\ .$$

2.3.4. Proposition. *If* $\alpha \in A\otimes \bar{A}^{\otimes 2p}\otimes A$, *we have*:

(a) $\alpha^{(m)} \square \alpha^{(n)} = \binom{n+m-1}{n-1}\alpha^{(n+m)}$,

(b) $\alpha^{(m)} * \alpha^{(n)} = \binom{n+m}{n}\alpha^{(n+m)}$.

Proof. (a) It will be done by induction on $n+m$. For $n+m=1$, it is obvious. Now, using Remark 2.3.2(b), we have:

$$\alpha^{(m+1)} \square \alpha^{(n)} = (\alpha \square \alpha^{(m)}) \square \alpha^{(n)} = \alpha \square (\alpha^{(m)} \square \alpha^{(n)}) + \alpha \square (\alpha^{(n)} \square \alpha^{(m)})\ .$$

By the inductive hypothesis,

$$\begin{aligned}\alpha^{(m+1)} \square \alpha^{(n)} &= \left(\binom{n+m-1}{m-1} + \binom{n+m-1}{n-1}\right)\alpha^{(n+m+1)} \\ &= \left(\binom{n+m-1}{m-1} + \binom{n+m-1}{m}\right)\alpha^{(n+m+1)} \\ &= \binom{n+m}{m}\alpha^{(n+m+1)} .\end{aligned}$$

(b) It follows by direct computation because

$$\begin{aligned}\alpha^{(m)} * \alpha^{(n)} &= \alpha^{(m)} \square \alpha^{(n)} + \alpha^{(n)} \square \alpha^{(m)} \\ &= \left(\binom{n+m-1}{m-1} + \binom{n+m-1}{m}\right)\alpha^{(n+m)} \\ &= \binom{n+m}{m}\alpha^{(n+m)} . \qquad \square\end{aligned}$$

2.3.5. Remark. The product in the normalized Hochschild complex $(A \otimes \bar{A}^{\otimes^*}, b)$ induced by $\square$, will be denoted by the same symbol $\square$. This product has the analogous properties.

2.3.6. Definition. Let $h_* : R(A) \to (A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ be the A^e-algebra map defined by:

$$\begin{aligned}&h_0 = \mathrm{id} , \\ &h_1(e_i) = -1 \otimes X_i \otimes 1 && (1 \le i \le n) , \\ &h_2(t_j) = \varepsilon_0\Big(-\sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot (1 \otimes X_i \otimes 1)\Big) && (1 \le j \le r) , \\ &h_{2p}(t_j^{(p)}) = (h_2(t_j))^{(p)} && (1 \le j \le r) .\end{aligned}$$

To prove that h_* is an A^e-algebra map it is enough to compare formula (b) of Proposition 2.3.4 with (1) of Section 2.2.

Since both d_* and b' are derivations, it is sufficient to prove that $h_{*-1}d_* = b'h_*$ for the generators e_i and $t_j^{(p)}$. In fact,

$$b'h_1(e_i) = b'(-1 \otimes X_i \otimes 1) = T(X_i) = d_1(e_i) = h_0 d_1(e_i) .$$

To prove the result for $t_j^{(p)}$ we use induction on p. The case $p=0$ is trivial. Using Remark 2.3.2(c), we have:

$$\begin{aligned}\varepsilon_0 h_{2p-1} d_{2p}(t_j^{(p)}) &= \varepsilon_0 h_{2p-1}\Big(\sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot e_i t_j^{(p-1)}\Big) \\ &= \varepsilon_0\Big(\Big(-\sum_{i=1}^{n} \frac{T_i(f_j)}{T(X_i)} \cdot (1 \otimes X_i \otimes 1)\Big) * (h_2(t_j))^{(p-1)}\Big) \\ &= h_2(t_j) \square (h_2(t_j))^{(p-1)} = (h_2(t_j))^{(p)} = h_{2p}(t_j^{(p)}) .\end{aligned}$$

We already know by the inductive hypothesis that

$$b'h_{2p-1}d_{2p}(t_j^{(p)}) = h_{2p-2}d_{2p-1}d_{2p}(t_j^{(p)}) = 0\,.$$

So,

$$\begin{aligned} b'h_{2p}(t_j^{(p)}) &= b'\varepsilon_0 h_{2p-1}d_{2p}(t_j^{(p)}) \\ &= (\mathrm{id} - \varepsilon_0 b')h_{2p-1}d_{2p}(t_j^{(p)}) = h_{2p-1}d_{2p}(t_j^{(p)})\,. \end{aligned}$$

3. Computation of Hochschild homology and cohomology

3.1. Introduction

Let $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r\rangle$, where k is an arbitrary commutative ring with 1, and $f_1, \ldots, f_r$ is a regular sequence in $k[X_1, \ldots, X_n]$. Assuming that A is k-flat, we compute the Hochschild homology $H_*(A, M)$ of A with coefficients in an A-module M, considered as an A^e-module by the multiplication $\mu : A^e \to A$. If we add the hypothesis that A is k-projective, we obtain the Hochschild cohomology $H^*(A, M)$.

3.2. Computation of Hochschild homology and cohomology

By tensoring the resolution $R(A)$ with M over A^e, using the identification $M \otimes_{A^e} (A \otimes_K A) \simeq M$ under which $(a \otimes a')m = aa'm$, we obtain the complex

$$\bar{R}(M):\ \cdots \longrightarrow \bar{M}_m \xrightarrow{\bar{d}_{m-1}} \bar{M}_{m-1} \xrightarrow{\bar{d}_{m-2}} \cdots \longrightarrow \bar{M}_1 \xrightarrow{\bar{d}_0} \bar{M}_0\,,$$

where $\bar{M}_m$ is the direct sum of copies of M indexed by $e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)}$ for $1 \le i_1 < \cdots < i_s \le n$; $s, p_1, \ldots, p_r \ge 0$ and $s + 2(p_1 + \cdots + p_r) = m$ and (with the convention of $t_j^{(p)} = 0$ if $p < 0$)

$$\begin{aligned} &\bar{d}_{m-1}(\vartheta \cdot e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)}) \\ &\quad = \sum_{j=1}^{r} \vartheta \cdot df_j \wedge e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_j^{(p_j-1)} \cdots t_r^{(p_r)}\,, \end{aligned}$$

with

$$df_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} \cdot e_i \quad \text{and} \quad \vartheta \in M\,.$$

The morphism $h_* : R(A) \to (A \otimes \bar{A}^{\otimes^*} \otimes A, b')$ defined in Section 2.3, induces a quasi-isomorphism $\bar{h}_* : \bar{R}(A) \to (A \otimes \bar{A}^{\otimes^*}, b)$ of differential graded strictly anti-commutative algebras, given by:

$$\bar{h}_m : \bar{A}_m \to A \otimes \bar{A}^{\otimes m},$$
$$\bar{h}_m(e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)})$$
$$= (-1)^s (1 \otimes X_{i_1}) * (1 \otimes X_{i_2}) * \cdots * (1 \otimes X_{i_s})$$
$$* (\bar{h}_2(t_1))^{(p_1)} * \cdots * (\bar{h}_2(t_r))^{(p_r)},$$

where, if $f_j = \sum f_{j i_1 \cdots i_n} X_1^{i_1} \cdots X_n^{i_n}$, then

$$\bar{h}_2(t_j) = \sum_{i_1,\ldots,i_n} \sum_{s=1}^{n} \sum_{k=0}^{i_s - 1} f_{j i_1 \cdots i_n} \cdot X_s^{i_s - k - 1} \cdot X_{s+1}^{i_{s+1}} \cdots X_n^{i_n}$$
$$\otimes X_1^{i_1} \cdots X_{s-1}^{i_{s-1}} \cdot X_s^k \otimes X_s .$$

3.2.1. Remark. $\bar{R}(M) = \bigoplus_{p \geq 0} \bar{R}^p(M)$, where

$$\bar{R}^p(M)\colon\ 0 \to \bar{M}_{2p}^p \xrightarrow{\bar{d}_{2p-1}^p} \bar{M}_{2p-1}^p \xrightarrow{\bar{d}_{2p-2}^p} \cdots$$
$$\xrightarrow{\bar{d}_{2p-n+1}^p} \bar{M}_{2p-n+1}^p \xrightarrow{\bar{d}_{2p-n}^p} \bar{M}_{2p-n}^p \to 0,$$

where

$$\bar{M}_{2p-s}^p = \bigoplus_{\gamma_{2p-s,p}} M^{\binom{n}{s}} t_1^{(p_1)} \cdots t_r^{(p_r)},$$

with $\gamma_{2p-s,p} = \{p_1, \ldots, p_r \geq 0\colon p_1 + \cdots + p_r = p - s\}$, and

$$\bar{d}_{2p-s}^p = \bar{d}_{2p-s}|_{\bar{M}_{2p-s}^p} .$$

In consequence, $H_m(A, M) = \bigoplus_{Um} H_m(\bar{R}^p(M))$, where $Um = \{p \in \mathbb{N}\colon (m/2) \leq p \leq \min(m, (m+n)/2)\}$.

3.2.2. Lemma. $H_m(\bar{R}^m(M)) = \Omega^m(A) \otimes_A M$ *for* $0 \leq m \leq n$.

Proof. It follows by an easy computation that

$$H_m(\bar{R}^m(M))$$
$$= \operatorname{coker}(\bar{M}_{m+1}^m \xrightarrow{\bar{d}_m^m} \bar{M}_m^m)$$
$$= \operatorname{coker}\Big(\bigoplus_{p_1+\cdots+p_r=1} M^{\binom{n}{m-1}} t_1^{(p_1)} \cdots t_r^{(p_r)} \xrightarrow{\bar{d}_m^m} M^{\binom{n}{m}}\Big)$$
$$= \operatorname{coker}\Big(\bigoplus_{p_1+\cdots+p_r=1} A^{\binom{n}{m-1}} t_1^{(p_1)} \cdots t_r^{(p_r)} \otimes_A M \xrightarrow{\bar{d}_m^m} A^{\binom{n}{m}} \otimes_A M\Big)$$
$$= \Big(\textstyle\bigwedge^{\binom{n}{m}} A^{(n)}\Big) \otimes_A M \xrightarrow{\bigwedge^{\binom{n}{m}}(\pi) \otimes 1_M} \Big(\textstyle\bigwedge^{\binom{n}{m}} A^{(n)} / \langle df_1, \ldots, df_r \rangle\Big) \otimes_A M ,$$

where

$$df_j = \sum_{i=1}^{n} \frac{\partial f_j}{\partial X_i} \cdot e_i$$

and $A^{(n)} \xrightarrow{\pi} A^{(n)}/\langle df_1, \ldots, df_r \rangle$ is the canonical projection. The proof is completed by noting that

$$\begin{aligned} \Omega^1(A) &= H_1(A) = H_1(\bar{R}^1(A)) \\ &= \operatorname{coker}\Big(\bigoplus_{p_1+\cdots+p_r=1} At_1^{(p_1)} \cdots t_r^{(p_r)} \xrightarrow{\bar{d}_m^m} A^{\binom{n}{1}} \Big) \\ &= A^{(n)}/\langle df_1, \ldots, df_r \rangle , \end{aligned}$$

because $\bar{d}_1^1(t_j) = df_j$. □

3.2.3. Notation. Let M be an A-module, $q \geq 0$ and

$$U = [U_{\alpha,\beta}]_{1 \leq \alpha \leq r, 1 \leq \beta \leq n}$$

a matrix with entries in A.

(a) We denote by $K^p(M, U)$ the *generalized Koszul complex* defined by

$$\begin{aligned} K^p(M, U)\colon\ 0 \longrightarrow K_{2p}^p \xrightarrow{d_{2p-1}^p} K_{2p-1}^p \xrightarrow{d_{2p-2}^p} \cdots \\ \xrightarrow{d_{2p-n+1}^p} K_{2p-n+1}^p \xrightarrow{d_{2p-n}^p} K_{2p-n}^p \to 0 \end{aligned}$$

where

$$K_{2p-s}^p = \bigoplus_{\gamma_{2p-s,p}} M^{\binom{n}{n-s}} t_1^{(p_1)} \cdots t_r^{(p_r)} ,$$

with $\gamma_{2p-s,p} = \{p_1, \ldots, p_r \geq 0\colon p_1 + \cdots + p_r = p - s\}$, and (with the convention of $t_j^{(p)} = 0$ if $p < 0$),

$$\begin{aligned} &\bar{d}_{2p-s}(\vartheta \cdot e_{i_1 \cdots i_{n-s+1}} t_1^{(p_1)} \cdots t_r^{(p_r)}) \\ &\quad = \sum_{j=1}^{r} \sum_{k=1}^{n-s+1} (-1)^{k+1} U_{j,i_k} \cdot \vartheta \cdot e_{i_1 \cdots \hat{i}_k \cdots i_{n-s-1}} t_1^{(p_1)} \cdots t_j^{(p_j-1)} \cdots t_r^{(p_r)} . \end{aligned}$$

(b) We denote by $K_p(M, U)$ the *generalized Koszul complex* defined by

$$\begin{aligned} K_p(M, U)\colon\ 0 \longrightarrow K_p^{2p} \xrightarrow{d_p^{2p-1}} K_p^{2p-1} \xrightarrow{d_p^{2p-2}} \cdots \\ \xrightarrow{d_p^{2p-n+1}} K_p^{2p-n+1} \xrightarrow{d_p^{2p-n}} K_p^{2p-n} \to 0 , \end{aligned}$$

where

$$K_p^{2p-s} = \bigoplus_{\gamma_{2p-s,p}} M^{(\binom{n}{s})} t_1^{(p_1)} \cdots t_r^{(p_r)} ,$$

with $\gamma_{2p-s,p} = \{p_1, \ldots, p_r \geq 0 \colon p_1 + \cdots + p_r = p - s\}$, and

$$\begin{aligned} &\bar{d}_p^{2p-s}(\vartheta \cdot e_{i_1 \cdots i_{s-1}} t_1^{(p_1)} \cdots t_r^{(p_r)}) \\ &\quad = \sum_{j=1}^{r} \sum_{k=1}^{s-1} U_{j,i_k} \cdot \vartheta \cdot e_{i_k} \wedge e_{i_1 \cdots i_{s-1}} t_1^{(p_1)} \cdots t_j^{(p_j+1)} \cdots t_r^{(p_r)} . \end{aligned}$$

3.2.4. Remark. If M is reflexive, $K_p(M, U)$ and $K^p(M, U)$ are the complexes $C_p(U)$ and $C_p^*(U)$ respectively from [2, Chapter 2].

3.2.5. Remark. Let $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$, where k is an arbitrary commutative ring with 1, and $f_1, \ldots, f_r$ is a regular sequence in $k[X_1, \ldots, X_n]$, M is an A-module and $U = [U_{\alpha,\beta}]_{1 \leq \alpha \leq r, 1 \leq \beta \leq n}$ is the matrix defined as $U_{\alpha,\beta} = (-1)^{\beta+1}(\partial f_\alpha / \partial X_\beta)$. The family $(\bar{M}_{2p-s}^p \xrightarrow{\varphi_{2p-s}} K_{2p-s}^p)$ is an isomorphism of $\bar{R}^p(M)$ in $K^p(M, U)$, where

$$\varphi_{2p-s}(\vartheta \cdot e_{i_1 \cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)}) = \vartheta \cdot e_{i'_1 \cdots i'_{n-s}} t_1^{(p_1)} \cdots t_r^{(p_r)} ,$$

with $1 \leq i'_1 < \cdots < i'_{n-s} \leq n$ and $\{i_1, \ldots, i_s\} \cap \{i'_1, \ldots, i'_{n-s}\} = \emptyset$.

3.2.6. Theorem. *Let $A = k[X_1, \ldots, X_n]/\langle f_1, \ldots, f_r \rangle$, where k is an arbitrary commutative ring with 1, and $f_1, \ldots, f_r$ is a regular sequence in $k[X_1, \ldots, X_n]$, M is an A-module and $U = [U_{\alpha,\beta}]_{1 \leq \alpha \leq r, 1 \leq \beta \leq n}$ is the matrix defined as $U_{\alpha,\beta} = (-1)^{\beta+1}(\partial f_\alpha / \partial X_\beta)$. Suppose that A is k-flat and regard M as an A^e-module using the morphism $\mu : A^e \to A$. We have*:

(1) *If $m \leq n$,*

$$\begin{aligned} H_m(M, A) &= [\bigoplus_{(m/2) \leq p < m} H_m(\bar{R}^p(M))] \oplus (\Omega^m(A) \otimes_A M) \\ &= [\bigoplus_{(m/2) \leq p < m} H_m(K^p(M, U))] \oplus (\Omega^m(A) \otimes_A M) . \end{aligned}$$

(2) *If $m > n$,*

$$\begin{aligned} H_m(M, A) &= \bigoplus_{(m/2) \leq p \leq (m+n)/2} H_m(\bar{R}^p(M)) \\ &= \bigoplus_{(m/2) \leq p \leq (m+n)/2} H_m(K^p(M, U)) . \end{aligned}$$

(3) *The product defined in Section 1.2 over $H_*(A)$ is also induced by*

$$\bar{A}^{p}_{2p-s} \times \bar{A}^{p'}_{2p'-s'} \longrightarrow \bar{A}^{p+p'}_{2(p+p')-(s+s')},$$
$$(e_{i_1\cdots i_s} t_1^{(p_1)} \cdots t_r^{(p_r)}, e_{j_1\cdots j_{s'}} t_1^{(p'_1)} \cdots t_r^{(p'_r)})$$
$$\mapsto e_{i_1\cdots i_s} \wedge e_{j_1\cdots j_{s'}} t_1^{(p_1+p'_1)} \cdots t_r^{(p_r+p'_r)}.$$

Proof. Since A is k-flat $H_*(A, M) = \mathrm{Tor}_*^{A^e}(A, M)$. So, $H_*(A, M) = H_*(R(A) \otimes_{A^e} M) = H_*(\bar{R}(M))$. Thus, (1) and (2) follow immediately from Remark 3.2.1, Lemma 3.2.2 and Remark 3.2.5, and (3) is true because $\bar{h}_*$ is a quasi-isomorphism of differential graded strictly anti-commutative algebras. □

3.2.7. Theorem. *Let A and M be as in Theorem 3.2.6, and let $U = [U_{\alpha,\beta}]_{1\le\alpha\le r, 1\le\beta\le n}$ be the matrix given by $U_{\alpha,\beta} = (\partial f_\alpha / \partial X_\beta)$. Suppose that A is k-projective and regard M as an A^e-module through the multiplication map $\mu : A^e \to A$. We have:*

(1) *If $m \le n$,*

$$H^m(M, A) = \bigoplus_{(m/2)\le p\le m} H^m(K_p(M, U)).$$

(2) *If $m > n$,*

$$H^m(M, A) = \bigoplus_{(m/2)\le p\le (m+n)/2} H^m(K_p(M, U)).$$

Proof. $(A \otimes \bar{A}^{\otimes^*} \otimes A)$ is a projective resolution of A as an A^e-module, so $H^m(A, M) = \mathrm{Ext}^m_{A^e}(A, M) = H^m(\mathrm{Hom}_{A^e}(R(A), M))$. On the other hand, M and $\mathrm{Hom}_A(A, M)$ are isomorphic A^e-modules, where the action of A^e is defined by $m(a \otimes b) = abm$ and $[(a \otimes b)f](x) = f(xab)$, respectively. Then,

$$\begin{aligned} H^m(A, M) &= H^m(\mathrm{Hom}_{A^e}(R(A), M)) \\ &= H^m(\mathrm{Hom}_{A^e}(R(A), \mathrm{Hom}_A(A, M)) \\ &= H^m(\mathrm{Hom}_A(R(A) \otimes_{A^e} A, M)) = H^m(\mathrm{Hom}_A(\bar{R}(A), M) \\ &= \bigoplus_{p\ge 0} H^m(\mathrm{Hom}_A(\bar{R}^p(A), M)). \end{aligned}$$

The proof is completed by noting that $\mathrm{Hom}_A(\bar{R}^p(A), M)$ is isomorphic to the complex $K_p(M, U)$. □

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